The Derivatives of Exponential Functions (Section 3.2)

Date: ______________

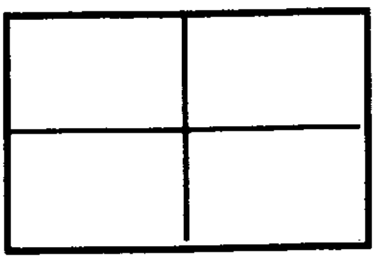
Quick note about logs. We know: $\ln 1 =$ ________, $\ln e =$ ________, $\ln 2 =$ ________

Today we want to discover the derivatives of exponential functions. So, what do we expect the derivative of a function in the form $f(x) = a^x$ to look like?

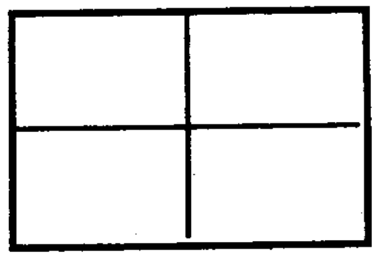
First, we know $f(x) = a^x$ has two different basic shapes. One, if $a > 1$, and the other if $0 < a < 1$.

Sketch the two graphs below.

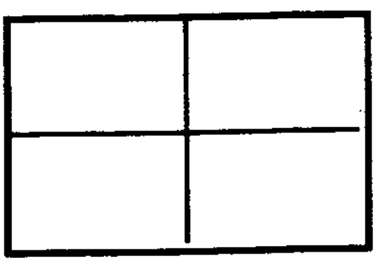
$a > 1$

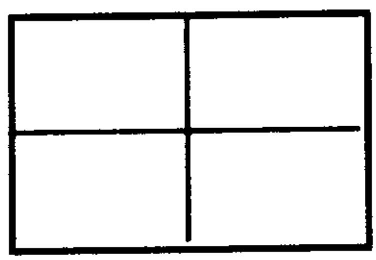


$0 < a < 1$



Sketch the graph of the derivative function of each of the above.





So, if $a > 1$, the graph of f' resembles the graph of f itself! Is it the same? Let's see by examining the graphs of $f(x) = 2^x$ and $f(x) = 3^x$, and their derivatives.

As we have done before, in your calculator enter:

$Y1 = 2^X$ and $Y2 = \text{nDeriv}(Y1, X, X)$

What are your observations? ____

Now change Y1 to $Y1 = 3^X$. (We don't have to change Y2.)

What are your observations? ____

This leads to two important questions.

1. Is there a number between 2 and 3 for the base of the exponential function that will make the graph of the function f and the function f' match exactly?

2. What are the exact derivatives of $f(x) = 2^x$ and $f(x) = 3^x$?

Let's look at question #1 first.

1. Is there a number between 2 and 3 for the base of the exponential function that will make the graph of the function f and the function f' match exactly?

What do you think?

Conclusion: If $f(x) =$ __________ then $f'(x) =$ __________

What does this say about the graph of $f(x) = e^x$? ____________________

____

Now question #2.

2. What is the exact derivatives of $f(x) = 2^x$ and $f(x) = 3^x$?

Again, let $Y1 = 2^X$. (Keep Y2 = nDeriv(Y1 , X , X))

It appears that $\frac{d}{dx}(2^x) =$ _______________ TRACE on the derivative function to find the constant k.

It seems that k = __________. We have seen this number before. Where? _______________

Conclusion: $\frac{d}{dx}(2^x) =$ _______________

Make a conjecture about the derivative of $f(x) = 3^x$. $\frac{d}{dx}(3^x) =$ _______________ Try it!

What is the derivative of $f(x) = \left(\frac{1}{2}\right)^x$? $\frac{d}{dx}\left(\left(\frac{1}{2}\right)^x\right) =$ ____________________ Try it

So, in general:

The Exponential Rule

For any positive constant a, $\frac{d}{dx}(a^x) =$ ____________________

And again, as a special case of the Exponential Rule,

$\frac{d}{dx}(e^x) =$ ____________________

Examples:

1. If $f(x)=\frac{5^x}{3}-9e^x$, find $f'(x)$.

2. Find the equation of the tangent line to the graph of $f(x)=4\left(\frac{1}{3}\right)^x$ at $x=0$. Check your answer by graphing f and the tangent line.

3. Once again, the **Cliff Notes** problem.

We found the function that solved the Cliff Notes problem was $P(t)=4000(1.226)^t$, where t represented the number of years after 1958 and P was the value of the company in that year. Find $P'(12)$ and interpret your answer.

The Derivative of the Natural Logarithmic Function (Section 3.2) **Date:** ______________

Before we begin our new topic of the "derivative of the natural logarithmic function", let's look at one more application problem of exponential decay.

Radioactive Decay: Radioactive substances decay by spontaneously emitting radiation. These substances decay at a rate proportional to to the mass of the substances. This means that the mass of the substance can be modeled by our exponential decay functions; either the <u>continuous growth model</u> $P(t) = P_0e^{kt}$ or the <u>discrete growth model</u> $P(t) = P_0a^t$. Physicists express the rate of decay in terms of **half-life,** the time required for half of any quantity to decay.

Example: The radioactive substance Bismuth-210 has a half-life of 5.0 days. A sample originally has a mass of 800 mg.

a. Find the <u>continuous growth formula</u> for the mass remaining after t days.

P(t) = ______________

b. Convert the <u>continuous growth model</u> to the corresponding <u>discrete growth model</u>. What is the "daily decay rate"?

P(t) = ______________

Decay rate: __________

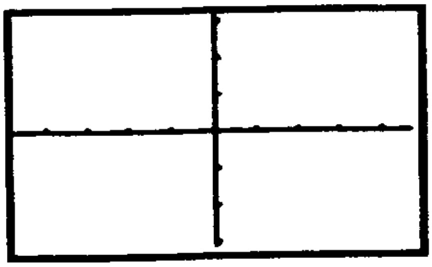
d. Use the <u>discrete growth model</u> (part b), and find the rate of change of the sample on the 5th day. On the 15th day.

Day 5: ____________

As we have done with many of our derivative properties, we will attempt a <u>graphical</u> approach in discovering the derivative of the natural logarithmic function, $f(x) = \ln x$.

Sketch the graph of $f(x) = \ln x$ below. (Can you do it without your calculator?)

f

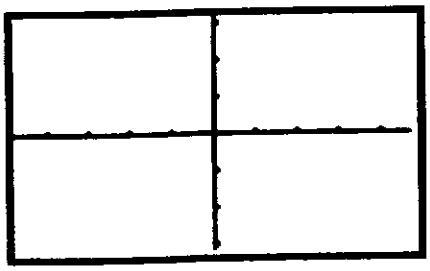


Label a few points that you know are on the graph of f.

By looking at the graph above, what information do you know about the derivative of the natural log function?

Use this information and sketch $f'(x)$, the derivative of $f(x) = \ln x$.

f'



Are we able to make a guess as to the function that we just graphed? Hint: It's a function that we know!

Conjecture: If $f(x) = \ln x$, then $f'(x) =$ ________ Or, using Leibniz notation ____________________

Let's check this conjecture graphically.

Y1 = ln(x) Y2 = nDeriv(Y1 , X , X) Y3 = (Our guess)

Before we do any examples of this new derivative property, let's write a summary of the derivative properties we now know.

Assume f and g are functions, and a and k are constants.

1. $\frac{d}{dx}(k) =$ ______ 2. $\frac{d}{dx}(x) =$ ________ 3. $\frac{d}{dx}(k \cdot x) =$ ____________

4. $\frac{d}{dx}(x^n) =$ ______ 5. $\frac{d}{dx}(f(x) \pm g(x)) =$ ____________ 6. $\frac{d}{dx}(k \cdot g(x)) =$ ____________

7. $\frac{d}{dx}(e^x) =$ ______ 8. $\frac{d}{dx}(a^x) =$ ________ 9. $\frac{d}{dx}(\ln x) =$ ____________

Examples:

1. If $f(x)=\frac{A}{x^5}+Be^x-C\ln x$, find $f'(x)$.

Note: Don't forget that the point of all of the derivative formulas is to use them to solve problems!

2. Let $f(x) = \ln x$.

   a. Find $f'(1/2)$, $f'(2)$, and $f'(10)$.

   $f'(1/2) =$ ____________ $f'(2) =$ ________ $f'(10) =$ ________

   When you compare these values, what do they tell you about the graph of f?

   b. Find $f''(2)$. Does this answer confirm your answer above?

3. **Web TV.** The number of homes with access to the Internet by way of cable television between 1998 and 2005 can be modeled by the equation:

   $I(x) = -138.27 + 76.29 \ln x$, where $I(x)$ is measured in million homes x years after 1990.

   a. Find $I(12)$ and interpret your answer (with units).

   b. Find $I'(12)$ and interpret your answer (with units).

4. If \$1000 is invested in a bank at an annual interest rate of 4.5%, and the interest is compounded continuously, then the amount of money $M(t)$ in the account after t years is given by the equation

$$M(t) = 1000e^{0.045t}$$

a. Solve this equation for t in terms of M; i.e. find $t(M)$.

b. Use the answer to part a to determine how long it will take for the investment to triple in value.

c. Find $\left.\frac{dt}{dM}\right|_{M=1000}$ and interpret your answer (with units).

$$\left.\frac{dt}{dM}\right|_{M=1000} =$$

A New and Important Derivative Property - The Chain Rule (Section 3.3) Date: ______________

I. Review of Composition of Functions

1. If $f(x) = |x|$, $g(x) = 1 - x^2$, and h(x)=ln x, what is:

   a. f(g(5))= ________________________________

   b. $g\left(h\left(e^3\right)\right) =$ ________________________________

   c. f(g(x))= ________________________________

   d. g(h(x))= ________________________________

   e. h(g(x))= ________________________________

2. Look at this problem in reverse.

   a. If h(x)=f(g(x)) and $h(x) = 2e^{5x+1}$,

      what is f(x)= ________________________________

      g(x)= ________________________________

   b. If h(x)=f(g(x)) and $h(x) = \left(3x^2 + 1\right)^3$,

      what is f(x)= ________________________________

      g(x)= ________________________________

   c. If h(x)=f(g(x)) and $h(x) = \sqrt{\ln x}$,

      what is f(x)= ________________________________

      g(x)= ________________________________

II. Apply composition of functions to our calculus

1. Make a guess.

   a. We know: If $y = 2e^x$, then $y' =$ ______________________

   Guess: If $y = 2e^{5x+1}$, then $y' =$ ______________________

   b. We know: If $y = x^3$, then $y' =$ ______________________

   Guess: If $y = (3x^2+1)^3$, then $y' =$ ______________________

   c. We know: If $y = \sqrt{x}$, then $y' =$ ______________________

   Guess: If $y = \sqrt{\ln x}$, then $y' =$ ______________________

2. Checking the guesses.

   a. Check the guess for Part a above <u>graphically</u>.

   Let: Y1= $2e^{5x+1}$

   Y2= ______________________ (The guess)

   Y3= nDeriv(Y1,X,X)

   Deactivate Y1 and graph Y2 and Y3.

   Was the guess correct? ________

   The correct answer is: If $y=2e^{5x+1}$ then $y' =$ ______________________

   b. Check the guess for Part b above algebraically.

   If $y = (3x^2+1)^3$, expand the right side of the equation, then find the derivative.

$$\begin{aligned} y &= (3x^2+1)^3 \\ &= (3x^2+1)(3x^2+1)(3x^2+1) \\ &= (3x^2+1)(9x^4+6x^2+1) \\ &= 27x^6+27x^4+9x^2+1 \end{aligned}$$

$y' =$

c. If we see the pattern, correct the guess for Part c, and check the answer graphically.

If $y = \sqrt{\ln x}$, then y'=______________________________

Let: $Y1 = \sqrt{\ln x}$

Y2= _________________________ (New guess for y')

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

III. Conclusion and Generalizations

1. We saw that if $h(x) = 2e^{5x+1}$, then $h'(x) = 2e^{5x+1} \cdot 5 = 10e^{5x+1}$.

Therefore, if $y = e^{g(x)}$, then y' = ______________________________

2. We saw that if $h(x) = \left(3x^2+1\right)^3$, then $h'(x) = 3\left(3x^2+1\right)^2 \cdot 6x = 18x\left(3x^2+1\right)^2$

Therefore, if $y = (g(x))^n$, then y' = ______________________________

3. This property for finding derivatives of a "composition of functions" is called the **Chain Rule.**

One more time, in the general form, the **Chain Rule** says:

If y = f(g(x)), then y'= ______________________________

Using Leibniz notation,

If y = f(g(x)), let z = g(x). Then y = f(z), and $\frac{dy}{dx} =$ ______________________________

IV. Examples:

1. Find derivatives of the following functions

   a. $s = (4t-1)^5$

b. $y = \sqrt{4 - 9x^2}$

c. $f(x) = (\ln x)^4$

d. $y = \sqrt[3]{(e^x + x)^2}$

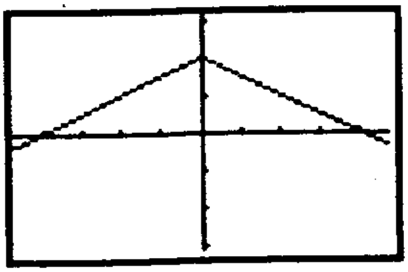
2. Suppose $5000 is deposited in a bank account that pays 4.5% annual interest, compounded continuously.

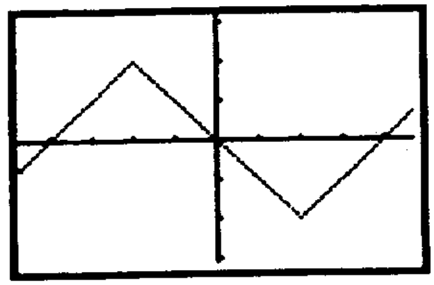
   a. Find a formula for P(t), the balance in the account t years after the initial deposit.

      P(t) = ______________________________

   b. Find P(10) and P'(10) and interpret your answers with units.

3. Use the figures below, to evaluate the expressions.

f(x) 

g(x) 

a. $g(3) =$ ________________

b. $f(-3) =$ ________________

c. $f(g(2)) =$ ________________

d. $g(f(1)) =$ ________________

e. $\left.\frac{d}{dx}f(x)\right|_{x=-2} =$ ________________

f. $\left.\frac{d}{dx}g(x)\right|_{x=4} =$ ________________

g. $\left.\frac{d}{dx}f(g(x))\right|_{x=1} =$ ________________

h. $\left.\frac{d}{dx}g(f(x))\right|_{x=-1} =$ ________________

Two New Derivative Properties (**Section 3.4**) **Date:** ________
The Product Rule and the Quotient Rule

I. The Product Rule

Introduction: We already have a property that allows us to find the derivative of a **sum** (or difference) of two functions.

Example:
If $f(x) = x^3 + x^5$,
then $f'(x) =$ ________ + ________

Property:
If $f(x) = g(x) + h(x)$,
then $f'(x) =$ ________ + ________

Objective: We want to discover a property that can be applied to find the **product** of two functions.

Let $g(x) = x^3$ and $h(x) = x^5$. Now, let $f(x) = g(x) \cdot h(x) =$ ________ • ________ = ________.

Find: $f'(x) =$ ________ $g'(x) =$ ________

$h'(x) =$ ________

$g'(x) \cdot h'(x) =$ ________

So, notice that although it is true that : **If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$,**

it is not true that: **If $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h'(x)$.**

It is possible to get the correct answer for $f'(x)$ by a clever combination of the equations for $g(x)$, $h(x)$, $g'(x)$, and $h'(x)$.

Write the functions again.

$g(x) =$ ______, $h(x) =$ ______, $f(x) =$ ______, $g'(x) =$ ______, $h'(x) =$ ______, $f'(x) =$ ______

(Remember, we are trying to get the function $f'(x)$ from a combination of $g(x)$, $h(x)$, $g'(x)$, and $h'(x)$.)

See if you can figure out what this combination is (with the following hints)!

Notice that the **8** in $f'(x) = 8x^7$ is the sum of the **3** and **5** in $g'(x) = 3x^2$ and $h'(x) = 5x^4$.

Fill in the following

$f'(x) = 8x^7 = 5x^7 + 3x^7 =$ ________ $\cdot 5x^4 +$ ________ $\cdot 3x^2$.

Notice what the functions are that you put in the blanks!

Now try to complete the conjecture. This derivative property is called the **Product Rule.**

Product Rule: If $f(x) = g(x) \cdot h(x)$, then $f'(x) =$ ________ · ________ + ________ · ________

When you think you have it, try the following. Assume that your conjecture is true for the product of any two functions.

If $f(x) = f(x) = x^2 \cdot \ln x$, then $f'(x) =$ ________ $\cdot$ ________ $+$ ________ $\cdot$ ________.

$=$ ________________________

or ________________________

Check your result graphically by graphing f(x) in Y1, your derivative in Y2, and the nDeriv(Y1,X,X) in Y3.

Try one more.

If $f(x) = x^3 e^{4x}$, then $f'(x) =$ ________ $\cdot$ ________ $+$ ________ $\cdot$ ________.

$=$ ________________________

or ________________________

Again, you can check your result graphically.

Examples of the Product Rule:

1. Let $f(x) = (3x^2 + 4)(2x^2 + 3)$

   a. Find $f'(x)$ by using the Product Rule.

   b. Find $f'(x)$ by expanding the terms first, then applying the Power Rule.

   c. Show that the two answers are equivalent.

2. Let $f(x) = x^2(x+3)^2$

a. Graph f on your calculator.

b. From the graph of f, for what x-values does it appear that f has a horizontal tangent?

x= ______________________________

c. Analytically, find all values of x where the graph of f has a horizontal tangent line.

3. If $f(x) = x \cdot e^x$, find $f'(x)$, $f''(x)$, $f'''(x)$, and a rule for $f^{(n)}(x)$, the nth derivative of f(x).

$f'(x) =$ ______________________________

$f''(x) =$ ______________________________

$f'''(x) = f^{(3)}(x) =$ ______________________________

$f^{(n)}(x) =$ ______________________________

Note: So, the above example shows that it's necessary to use the Product Rule to find the derivative of $f(x) = x \cdot e^x$. Is it necessary to use the Product Rule to find the derivative of $f(x) = 8 \cdot e^x$? Why or why not?

Could you use the Product Rule to find the derivative of $f(x) = 8 \cdot e^x$?

II. The Quotient Rule

New Objective: We want to discover a property that can be applied to find the **quotient** of two functions.

Actually, in many instances, the property is not necessary. For example, if we wanted to find the derivative, f'(x). for $f(x)=\frac{2^x}{x^3}$, we could rewrite the function as $f(x)=\frac{2^x}{x^3}=$ ____________________ and then find the derivative using the Product Rule. Lets do this!

f'(x) =

However, we will see that most of the time it is easier to leave the function as a quotient, and find the derivative without rewriting it.

Let's see if we can discover this property by using the preceding example. Follow the steps below, look at the fi[illegible] answer, and see if you can see a pattern!

Let $f(x)=\frac{2^x}{x^3}$.

So, if $f(x) = \frac{2^x}{x^3}$, then $f'(x) =$ ________________________________

(Note: Is this answer the same as our answer when we did the problem by rewriting the function and using the Product Rule?)

The result of this problem is the property called the **Quotient Rule**. The **Quotient Rule** says that:

If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) =$ ____________________________

It's time to practice!

Examples of the Quotient Rule:

1. If $f(x) = \frac{5x^2}{3x+5}$, find $f'(x)$.

2. Let $f(x)=\dfrac{x}{x^2+1}$.

a. Graph the function on your calculator and determine intervals where the function is increasing and decreasing.

Increasing ____________________ Decreasing ____________________

b. To verify the intervals above, find:

$f'(-2)=$ _____ $f'(-1)=$ _____ $f'(0)=$ _____ $f'(1)=$ _____ $f'(2)=$ _____

3. **Practice Problem**

Show that the derivative of $f(x)=\dfrac{ax+b}{cx+d}$ is $f'(x)=\dfrac{ad-bc}{(cx+d)^2}$, where a, b, c, and d are constants.

4. Write the equation of the tangent line to the function $y=\dfrac{e^x}{e^x+1}$ at $x=0$. Verify your answer graphically.

5. Given $f(x) = \dfrac{k}{g(x)}$. Find the derivative $f'(x)$ two different ways.

a. By using algebra to rewrite the function (using a negative exponent), then differentiating.

$f(x) = \dfrac{k}{g(x)} =$ ______________

b. By using the Quotient Rule.

6. Given $f(x) = \ln\left(\dfrac{x-1}{x+1}\right)$. Find the derivative $f'(x)$ two different ways.

a. By applying calculus right away, then simplifying the answer.

b. By simplifying the expression first, then applying calculus.

$f(x) = \ln\left(\dfrac{x-1}{x+1}\right) =$ ______________________________

c. Show that the answers for parts a and b above are equivalent.